MATH 5C - TEST 2 v1 Spring 24
(Chapter 14)

Instructions on Canvas
100 POINTS

(1) Given $f(x,y,z) = y\ln(z) + x^3\cos(xz^2)$, find the first order partial derivatives

(9 points)

$f_x(x,y,z) = 3x^2\cos(xz^2) - x^3z^2\sin(xz^2)$

$f_y(x,y,z) = \ln z$

$f(x,y,z) = \frac{y}{z} - 2x^4z\sin(xz^2)$

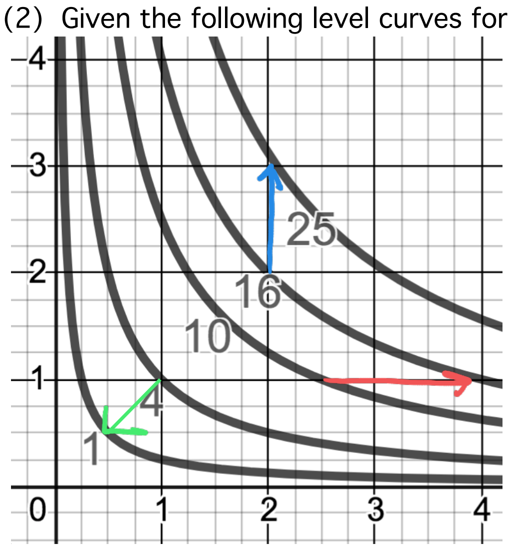
$f = ye^{xy} + x^3\cos(z^2)$

$f_x = y^2e^{xy} + 3x^2\cos(z^2)$

$f_y = e^{xy} + xye^{xy}$

$f_z = -2x^3z\sin(z^2)$

(2) Given the following level curves for an unknown function f(x,y),



Estimate the following. Show computations on b,c,d. (8 points)

(a) $f(1,1)$ ____4____

(b) $\left.\frac{\partial f}{\partial x}\right|_{(2.5,1)}$ ____4____

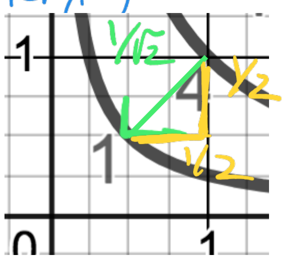
$\frac{16-10}{3/2} = 6 \cdot \frac{2}{3}$

(c) $f_y(2,2)$ ____9____

$\frac{25-16}{1}$ $\frac{\text{change in output}}{\text{change in input (or length)}}$

(d) $D_{\vec{u}}f(1,1)$ where $\vec{u} = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$ ____ $-3\sqrt{2}$

$\frac{1-4}{1/\sqrt{2}}$ ← length



(Accept any length between 1/2 and 1)

(3) Find all critical points of $f(x,y) = 6x - (x+3)^2 + 12y - y^3 + 15$ and classify each as yielding local max., local min., or saddle points. Show how you arrive at your conclusions. You are not required to find the functional values at the critical points.

(10 points)

(The graph is shown to help you see if your answer is reasonable)

Critical Points

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 6 - 2(x+3) = 0 & x = 0 \\ 12 - 3y^2 = 0 & y = \pm 2 \end{cases} \Rightarrow (0,2)(0,-2)$$

Apply Second Derivative Test

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -6y \end{vmatrix} = 12y$$

same

$D(0,2) = 24 > 0$ with $f_{xx} < 0 \Rightarrow$ local max at $(0,2)$

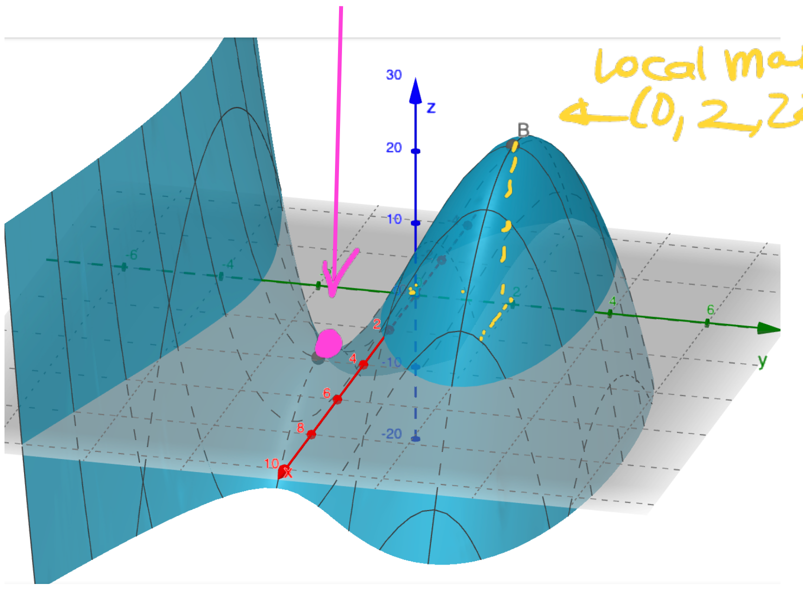
$D(0,-2) = -24 < 0 \Rightarrow$ saddle point at $(0,-2)$

$f(0,-2) = -9 - 24 + 8 + 15$
$= -10$

$f(0,2) = -9 + 24 - 8 + 15 = 22$

Saddle $(0,-2,-10)$

Local max
$\leftarrow (0,2,22)$



(4) Find the equation of the tangent plane to the ellipsoid $x^2 + y^2 + 2z^2 = 27$ at the point $(3, 4, 1)$.

(10 points)

Explain

Treat the ellipsoid as the level curve $F(x,y,z) = 27$, where $F(x,y,z) = x^2+y^2+2z^2$

$(\vec{\nabla} F = \langle 2x, 2y, 4z \rangle)$

Then $\vec{\nabla} F(3,4,1)$ will be orthogonal to that surface and can be used as the normal to the tangent plane

Plane:

point: $(3,4,1)$

$\vec{n} : \vec{\nabla} F(3,4,1) = \langle 6,8,4 \rangle$

$$6(x-3)+8(y-4)+4(z-1)=0$$

$$6x+8y+4z-52=0$$

$$3x+4y+2z-26=0$$

(5) Given $f(x,y) = ye^{xy}$, use differentials or a linear approximation to approximate the value of $f(0.2, 0.94)$ without using your calculator. (You can use your calculator to check your result). Show work clearly, labeling everything

(10 points)

Using $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$(a,b) = (0,1)$

$f_x(x,y) = y^2 e^{xy}$

$f_y(x,y) = xye^{xy} + e^{xy}$

$f(0,1) = 1$

$f_x(0,1) = 1$

$f_y(0,1) = 1$

$L(x,y) = 1 + 1(x-0) + 1(y-1)$

$L(x,y) = 1 + x + y - 1 = x + y$

$f(x,y) \approx L(x,y) = x + y$

$f(0.2, 0.96) \approx L(0.2, 0.94) = 0.2 + 0.9$

$= 1.14$

Notice - this can be computed easily - no calculator needed.

If you had to compute something like $e^{1/2}$ 0.94..., it wasn't a simple computation

(6) Find $\frac{\partial z}{\partial x}$, if z is defined implicitly as a function of x and y $\quad x^3+y^3+z^3+6xyz+4=0$

(8 points)

$$\frac{\partial}{\partial x}\left(x^3+y^3+z^3+6xyz+4\right)=\frac{\partial}{\partial x}(0)$$

$$3x^2+0+3z^2\frac{\partial z}{\partial x}+6y\left(z+x\frac{\partial z}{\partial x}\right)=0$$

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

$$\left(3z^2+6xy\right)\frac{\partial z}{\partial x}=-3x^2-6yz$$

$$\boxed{\frac{\partial z}{\partial x}=\frac{-3x^2-6yz}{3z^2+6xy}}$$

OR Let $F(x,y,z)=x^3+y^3+z^3+6xyz+4$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}$$

A

(7) Find the point(s) on the cone $z^2 = x^2 + y^2$ that are closest to (4,2,0)

(10 points)

Validate how you know it is an absolute minimum. (The second derivative test does NOT validate an absolute extrema)

(10 points)

Let $P(x,y,z)$ be any point on the cone

$$d(A,P) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

Subject to the point P must be on the cone so $z^2 = x^2 + y^2$

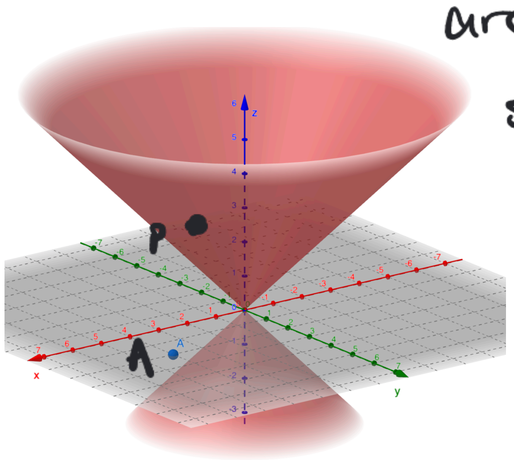
Minimize $d^2 = f(x,y) = (x-4)^2 + (y-2)^2 + x^2 + y^2$

Critical numbers: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\quad \begin{array}{l} 2(x-4) + 2x = 0 \\ 2(y-2) + 2y = 0 \end{array} \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$

$$z^2 = x^2 + y^2 = 2^2 + 1^2 = 5$$

$$z = \pm\sqrt{5}$$

Physically, we can see that we must have an absolute minimum distance. Both the critical points $(2, 1, \pm\sqrt{5})$ are equidistant to (4,2,0) so they must yield minimum



$(2, 1, \pm\sqrt{5})$

(8) Suppose you are at the point (2, 1, 82) on the hill given by $f(x,y)=100-4xy^2-2y-8$ (where the z axis gives elevation, the y axis points north, the x axis east). Correct notation is required, units are not needed

(13 points)

(a) If you hike from the given point in the direction of $\vec{v}=\langle 3,4\rangle$ are you going uphill or down? At what rate?

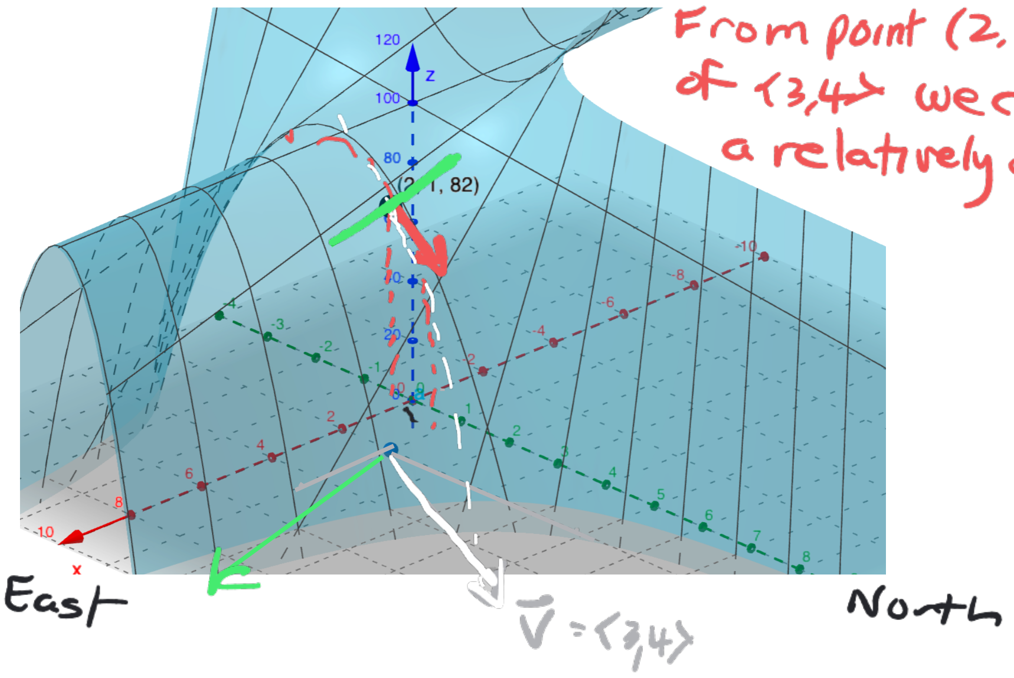
$\vec\nabla f(x,y)=\langle -4y^2, -8xy-2\rangle \qquad \vec u=\dfrac{\vec v}{\|\vec v\|}=\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

$\vec\nabla f(2,1)=\langle -4, -18\rangle$

$D_{\vec u} f(2,1)=\vec\nabla f(2,1)\cdot\vec u=\langle -4,-18\rangle\cdot\left\langle \frac{3}{5},\frac{4}{5}\right\rangle=\dfrac{-84}{5}$

down

(b) Using the graph of $f(x,y)$ with the point (2,1,~~90~~ 82) shown, explain and illustrate why your answers above are reasonable (or not).



From point (2,1,81) in direction of $\langle 3,4\rangle$ we can see it is a relatively steep down hill

(c) From the given point , if a you want to hike on a level path where elevation does not change, what direction should you go? (give a vector) .

Need to travel orthogonal to $\vec\nabla f(2,1)=\langle -4,-18\rangle$

so in direction $\pm\langle 18,-4\rangle$

$\pm\langle 9,-2\rangle$

(9) Using the method of LAGRANGE MULTIPLIERS, find the absolute extrema of $f(x,y) = x^2 + 4y^3$ subject to the constraint $x^2 + 2y^2 = 1$. Show all points you considered in the process. (You may assume that an absolute max and an absolute min exist) The graph is given in order so you may check whether your answer is reasonable. Show the absolute max and min on the graph by labeling it on the graph. (10 points)

____

Show all points which you considered as possibilities for yielding extreme values

$\vec\nabla f = \lambda \vec\nabla g$, $g(x,y)=K$ $\Rightarrow \begin{cases} 2x = \lambda 2x \\ 12y^2 = \lambda 4y \\ x^2+2y^2=1 \end{cases}$

$2x = \lambda 2x \Rightarrow 2x(1-\lambda)=0 \Rightarrow x=0$ or $\lambda = 1$

$x=0$, system becomes

$\begin{cases} 0=0 \checkmark \\ 12y^2 = \lambda 4y \\ 0+2y^2=1 \end{cases}$

$(0, \frac{1}{\sqrt{2}})$ $(0, -\frac{1}{\sqrt{2}})$

$\lambda = 1$ system becomes

$\begin{cases} 2x=2x \checkmark \\ 12y^2=4y \Rightarrow 4y(3y-1)=0 \\ x^2+2y^2=1 \end{cases}$

$y=0$

$\{ x^2+2y^2=1$

$x=\pm 1$

$(\pm 1, 0)$

$y=\frac{1}{3}$

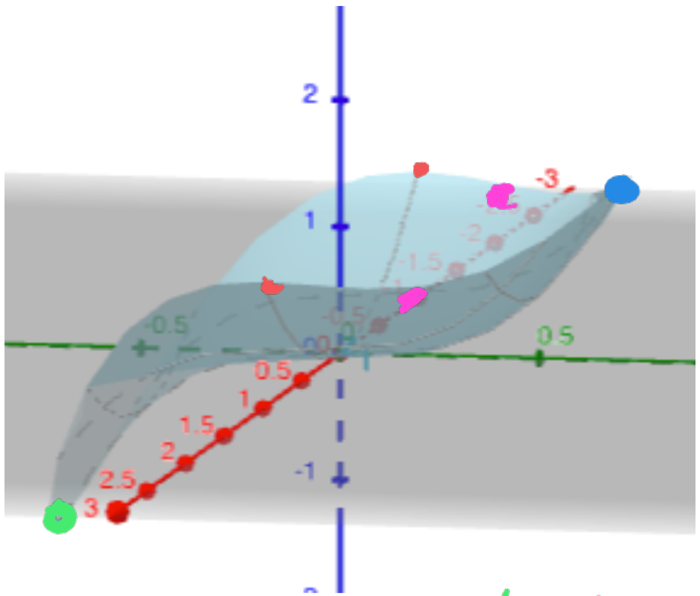
$x^2+2(\frac{1}{3})^2=1$

$x^2+\frac{2}{9}=1$

$x^2=\frac{7}{9}$

$x=\pm\frac{\sqrt{7}}{3}$

$(\pm\frac{\sqrt{7}}{3}, \frac{1}{3})$



(x,y)	$(0, \frac{1}{\sqrt{2}})$	$(0, -\frac{1}{\sqrt{2}})$	$(\pm 1, 0)$	$(\pm\frac{\sqrt{7}}{3}, \frac{1}{3})$
$F(x,y)$	$\sqrt{2}$	$-\sqrt{2}$	1	$\frac{25}{27}$

MAX is $\sqrt{2}$ MIN is $(-\sqrt{2})$

(10) Match the following functions with their graphs: (12 points) (axes are in standard orientation with the x axis in red, y green, z blue.)

a) $f(x,y)=\dfrac{1}{1+x^2y^2}$ __D______

(b) $f(x,y)=\cos(x+y)$ __A______

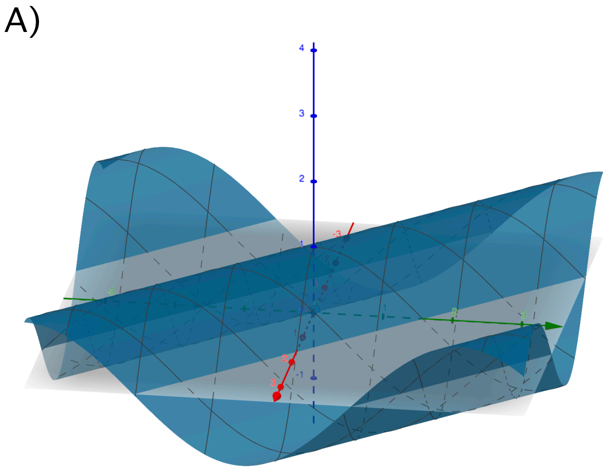
c) $f(x,y)=\ln(x^2-y)$ __F______

(d) $f(x,y)=\cos(xy)$ __B______

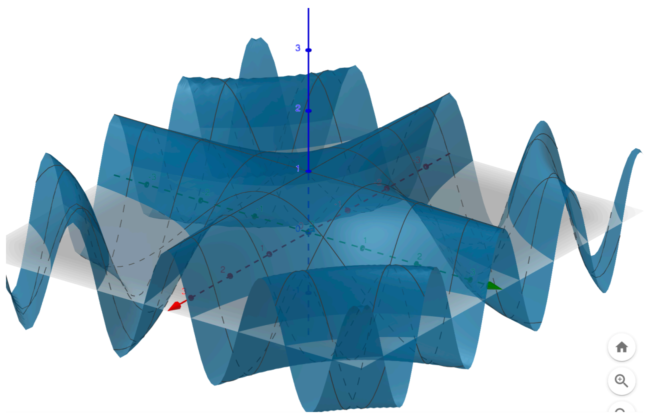
e) $f(x,y)=\dfrac{1}{x^2+y^2}$ __C______

(f) $f(x,y)=\dfrac{1}{x^2-y}$ __E______

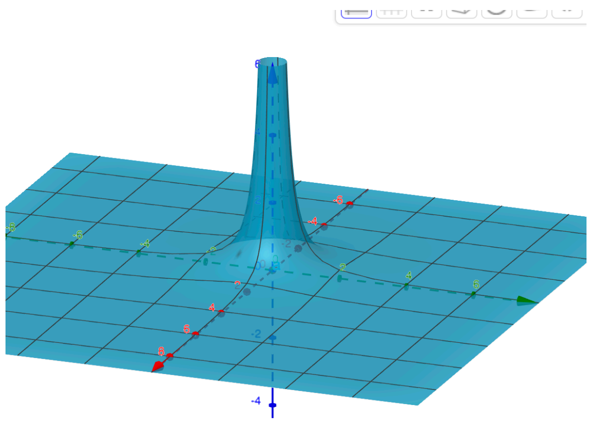
A)



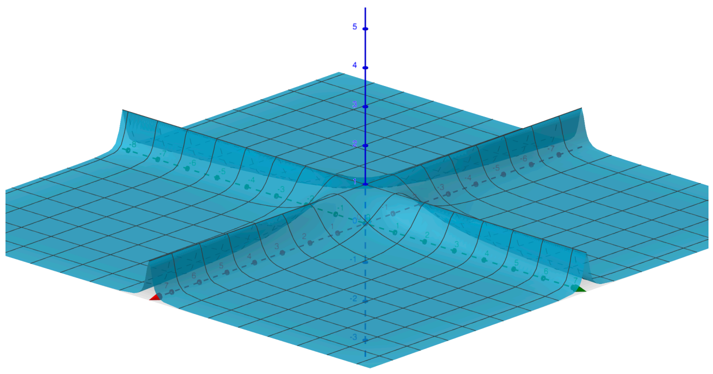
B)



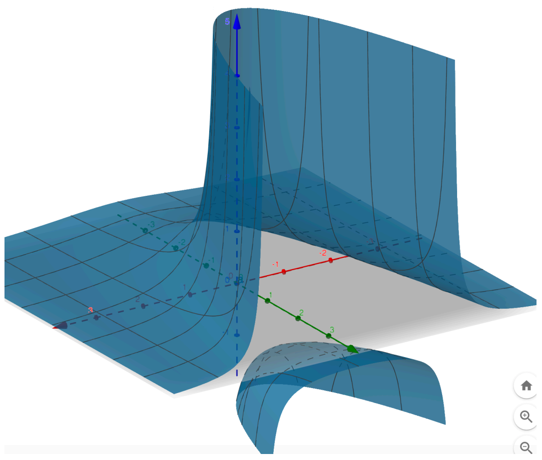
C)



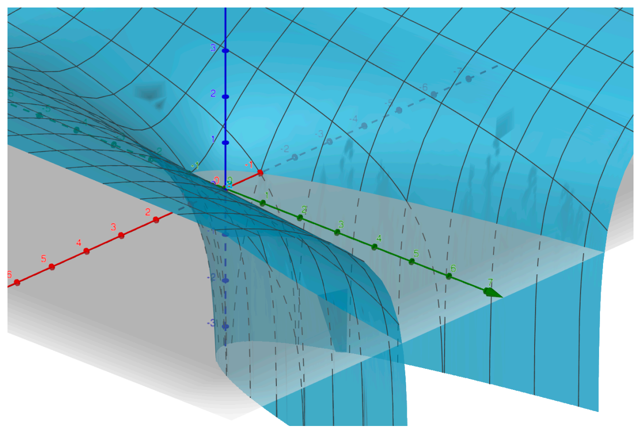
D)



E)



F)



MATH 5C - TEST 2 v2 Fall 2024
(Chapter 14)

Instructions on Canvas
100 POINTS

(1) Given $f(x,y,z) = xz\cos(z^2)+\dfrac{x}{y}$, find the first order partial derivatives .

Two versions of this problem

(9 points)

$f_x(x,y,z) = z\cos(z^2)+\frac{1}{y}$

$f_y(x,y,z) = \frac{-x}{y^2}$

$f_z(x,y,z) = -2xz\sin(z^2)+x\cos(z^2)$

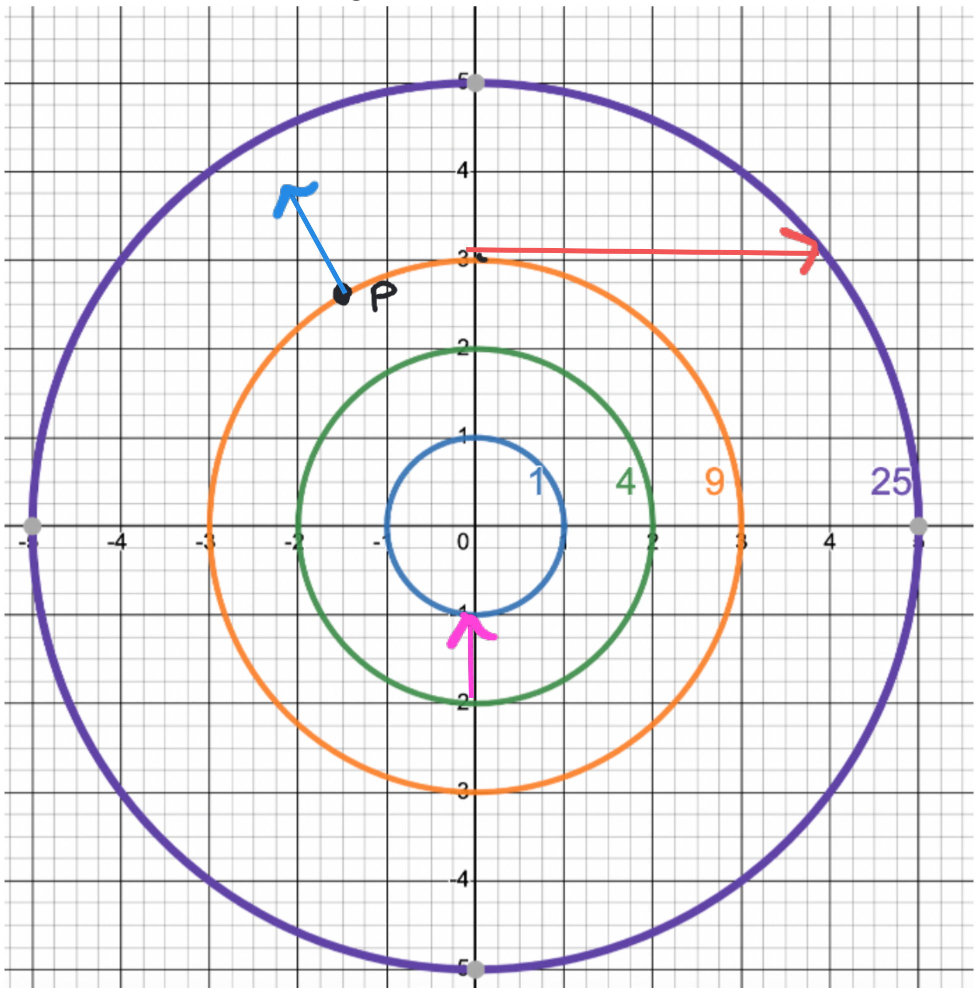
$f(x,y,z) = x\cos(z^2)+\frac{x}{y}$

$f_x = \cos(z^2)+\frac{1}{y}$

$f_y = -\frac{x}{y^2}$

$f_z = -2\cancel{x}z\sin(x^2)$

(2) Given the following level curves for an unknown function f(x,y),



Estimate the following. Show computations on b & c. (8 points)

(a) f(3,0) ___9___

(b) $\left.\dfrac{\partial f}{\partial x}\right|_{(0,3)}$ ___4___ $\dfrac{25-9}{4}$

(c) $f_y(0,-2)$ ___-3___ $\dfrac{1-4}{1}$

(d) On the figure above, sketch a vector in the direction of the gradient of f(x,y) at point P.

Orthogonal and pointing in direction of increase

(3) Find all critical points of $f(x,y) = -x^3 + 3x + y^3 - 3y^2$ and classify each as yielding local max., local min., or saddle points. Show how you arrive at your conclusions. You do not need to find the functional values at the critical points.

(10 points)

(The graph is given so you can see if your answer is reasonable, with the x axis in red, y green, z blue.)

Critical Points

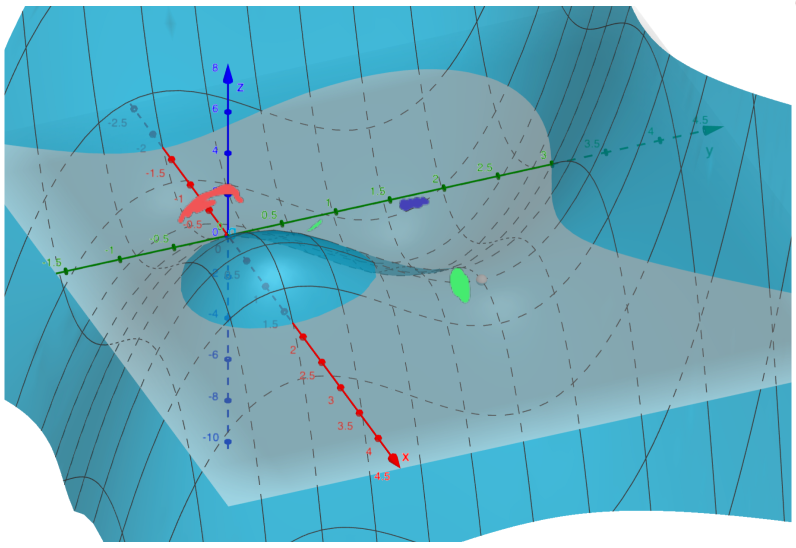
$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} -3x^2 + 3 = 0 \Rightarrow x = \pm 1 \\ 3y^2 - 6y = 0 \quad 3y(y-2) = 0 \quad y = 0, 2 \end{cases}$$

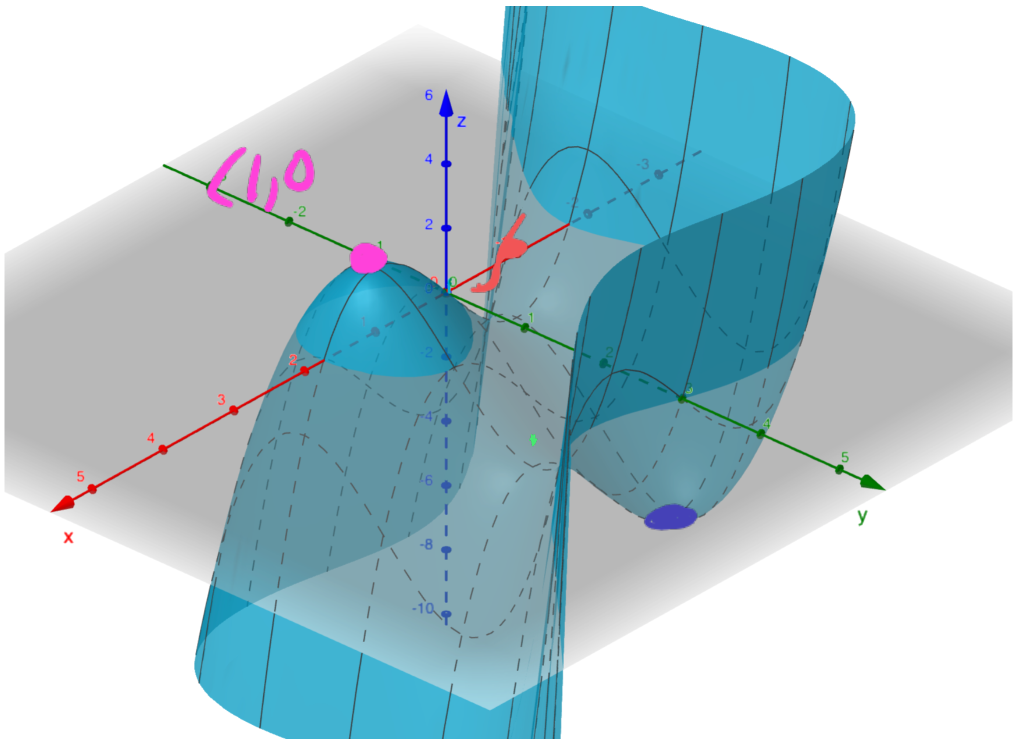
$(1,0)\ (1,2)$
$(-1,0)\ (-1,2)$

Apply Second Derivative Test

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6x & 0 \\ 0 & 6y-6 \end{vmatrix} = -36x(y-1)$$

crit point	$(1,0)$	$(1,2)$	$(-1,0)$	$(-1,2)$
D	36	-36	-36	36
f_{xx}	$-$			$+$
	Local Max	saddle	saddle (hard to tell) on graph	Local min





(4) Find the equation of the plane tangent to the surface $\sqrt{x^2 - yz} = 1$ at the point (3,2,4)

(10 points)

Explain

If $F(x,y,z) = \sqrt{x^2 - yz}$ then the given surface can be thought of as a level surface for F. We know that $\vec{\nabla} F$ gives a vector othogonal to a level surface so the normal to the tangent plane at (3,2,4) is

$$\vec{n} = \vec{\nabla} F(3,2,4)$$

$$F_x = \frac{x}{\sqrt{x^2 - yz}}$$

$$F_y = \frac{-z}{2\sqrt{x^2 - yz}}$$

$$F_z = \frac{-y}{\sqrt{x^2 - yz}}$$

$\vec{n} = \langle 3, -2, -1 \rangle$

point (3,2,4)

→ Plane

$$3(x-3) - 2(y-2) - (z-4) = 0$$

$$3x - 2y - z = 1$$

(5) Given $z = x^3 y - 3y^2$. If x changes from 2 to 1.9 while y changes from 1 to 1.02, find dz and Δz. Is your answer reasonable? Why? (10 points)

$$(2,1) \to (1.9, 1.02)$$

$$f_x = 3x^2 y$$
$$f_y = x^3 - 6y$$

$$\Delta z = f(1.9, 1.02) - f(2,1) \approx 3.87498 - 5 \approx -1.12502$$

$$dz = f_x(2,1)\Delta x + f_y(2,1)\Delta y$$
$$= 12(-.1) + 2(.02)$$
$$= -1.2 + .04 = -1.16$$

$dz \approx \Delta z$ as it should be for small $\Delta y, \Delta x$.

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(6) Find $\dfrac{\partial z}{\partial y}$: $\quad ye^{3x} + \sin(z) = 4z^3$ (8 points)

$$\frac{\partial}{\partial y}\left(ye^{3x} + \sin(z) \right) = \frac{\partial}{\partial y}\left(4z^3\right)$$
$$e^x + (\cos z)\frac{\partial z}{\partial y} = 12z^2 \frac{\partial z}{\partial y}$$
$$e^x = \frac{\partial z}{\partial y}\left(12z^2 - \cos z\right)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{e^x}{12z^2 - \cos z}}$$

OR

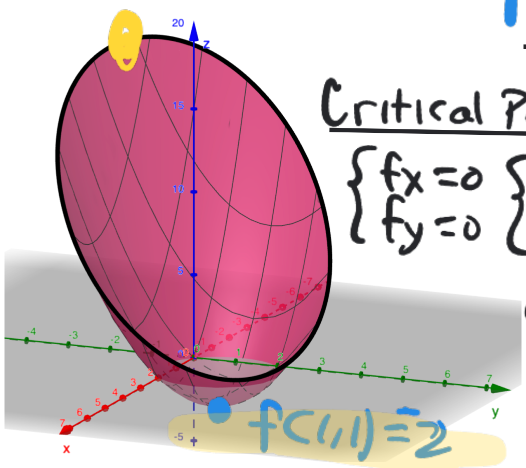
Using formula with $F(x,y,z) = ye^{3x} + \sin z - 4z^3$

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = \frac{-e^{3x}}{\cos z - 12z^2}$$

(7) Find the absolute max and min of $f(x,y) = x^2 + y^2 - 2x - 2y$ on the closed domain D given by $x^2 + y^2 \le 9$ (10 points)

Two versions of this problem

1

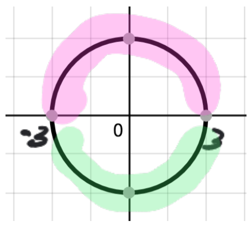


$f(1,1)=2$

(The graph is shown to see if your answer is reasonable)

Critical Points:

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\begin{cases} 2x-2=0 \\ 2y-2=0 \end{cases} \Rightarrow (1,1)$ $f(1,1) = -2$



On Boundary $x^2+y^2=9$

$y^2 = 9-x^2$

$y = \pm\sqrt{9-x^2}$

$f(x,y)$ becomes $g(x) = x^2 + 9 - x^2 - 2x - 2(\pm\sqrt{9-x^2})$

$g(x) = 9 - 2x - 2(\pm\sqrt{9-x^2})$

Top $g(x) = 9 - 2x - 2\sqrt{9-x^2}$

$g'(x) = -2 - \dfrac{2x}{\sqrt{9-x^2}} = 0$

$-2x = 2\sqrt{9-x^2}$

$4x^2 = 4(9-x^2)$

$2x^2 = 9$

$x = \pm\frac{3}{\sqrt{2}}$

Bottom $g(x) = 9 - 2x + 2\sqrt{9-x^2}$ $[-3,3]$

$g'(x) = -2 - \dfrac{2x}{\sqrt{9-x^2}} = 0$

$-2 = \dfrac{2x}{\sqrt{9-x^2}}$

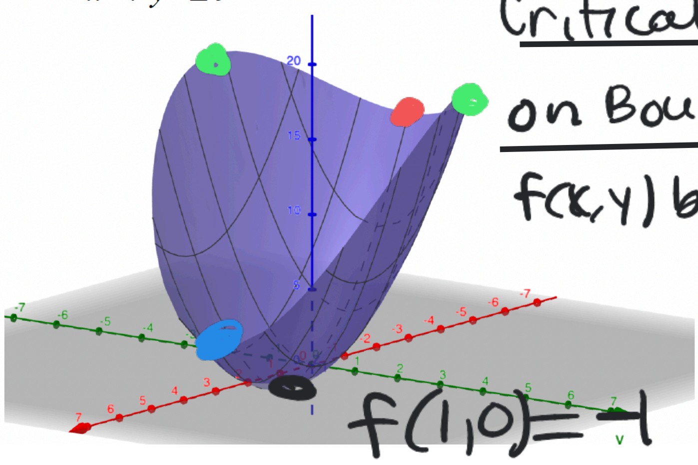
$x = \pm\frac{3}{\sqrt{2}}$

x	$(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$	$(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}})$	$(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$	$(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}})$	$(-3,0)$	$(3,0)$
$f(x)$	.515	9	9	17.485	15	3

ABS MIN -2 Abs MAX 17.485

2

(7) Find the absolute max and min of $f(x,y) = x^2 + 2y^2 - 2x$ on the closed domain D given by $x^2 + y^2 \le 9$ (10 points)



$f(1,0)=-1$

Critical Points $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x-2=0 \\ 4y=0 \end{cases}$ $(1,0)$

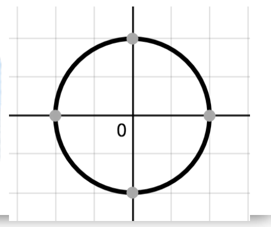
On Boundary $y^2 = 9 - x^2$

$f(x,y)$ becomes $g(x) = x^2 + 2(9-x^2) - 2x$

$g(x) = 18 - x^2 - 2x$ $[-3,3]$

$g'(x) = -2x - 2 = 0$ $x = -1$

x	-1	-3	3
$g(x)$	19	15	3



Max is 19, min is -1

8) Suppose you are at the point (2, 1, 82) on the hill given by $f(x,y)=100-4x^2-2y$ (where the z axis gives elevation, the y axis points north, the x axis east).Correct notation is required, units are not

(13 points)

(a) If you hike from the given point in the direction of $\vec{v}=\langle 3,4\rangle$ are you going uphill or down? At what rate?

$\vec{u}=\frac{1}{\|\vec{v}\|}\vec{v}=\langle \frac{3}{5},\frac{4}{5}\rangle$

$\vec{\nabla} f(x,y)=\langle -8x,-2\rangle$

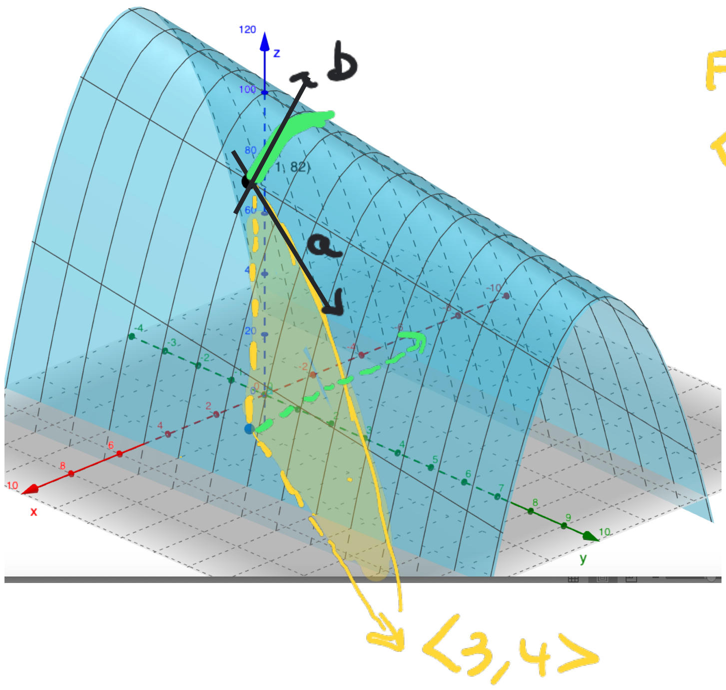
$\vec{\nabla} f(2,1)=\langle -16,-2\rangle$

$D_{\vec{u}}f(2,1)=\langle -16,-2\rangle\cdot\langle \frac{3}{5},\frac{4}{5}\rangle = \frac{-56}{5}$, down hill

(b) From the given point , in what direction is the steepest climb?

direction of gradient $\langle -16,-2\rangle$

(c) Using the graph of $f(x,y)$ with the point (2,1,82) shown, explain and illustrate why your answers above are reasonable (or not).



For part (a), direction $\langle 3,4\rangle$ means move in positive x, positive y direction

down hill

For part b, move in direction $\langle -16,-2\rangle$ so left (a lot) back (a little)

steep uphill

(9) Given that an absolute minimum exists, use the method of LAGRANGE MULTIPLIERS to find the absolute minimum of $f(x,y,z)=2x^2+y^2+3z^2$ subject to the constraint $2x-3y-4z=49$. (10 points)

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y,z)=k \end{cases}$$

$4x = \lambda 2$

$2y = \lambda(-3)$

$6z = \lambda(-4)$

$2x-3y-4z=49$

$$\begin{cases} x = \lambda/2 \\ y = -\frac{3\lambda}{2} \\ z = -\frac{4\lambda}{6} = -\frac{2\lambda}{3} \end{cases}$$

$$2\left(\frac{\lambda}{2}\right)-3\left(-\frac{3\lambda}{2}\right)-4\left(\frac{-2\lambda}{3}\right)=49$$

$$6\left(\lambda+\frac{9}{2}\lambda+\frac{8}{3}\lambda\right)=(49)6$$

$$6\lambda+27\lambda+16\lambda=49\cdot 6$$

$$49\lambda=(49)(6)$$

$$\lambda=6$$

$\Rightarrow$ $x=\frac{\lambda}{2}=3$ $\quad y=-\frac{3\lambda}{2}=-9$ $\quad z=\frac{-2\lambda}{3}=-4$

$(3,-9,-4)$

Only posibility and we were told min must exist so

Abs. Min of $f(3,-9,-4)=\boxed{147}$

Two versions

(10) Match the following functions with their graphs: (12 points) (axes are in standard orientation with the x axis in red, y green, z blue.)

a) $f(x,y)=\dfrac{1}{1+x^2y^2}$ __D__

(b) $f(x,y)=\cos(x+y)$ __A__

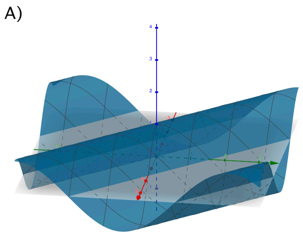
c) $f(x,y)=\ln(x^2-y)$ __F__

(d) $f(x,y)=\cos(xy)$ __B__

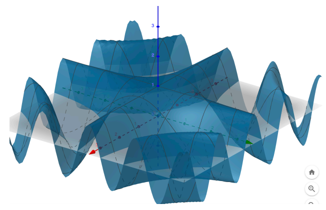
e) $f(x,y)=\dfrac{1}{x^2+y^2}$ __C__

(f) $f(x,y)=\dfrac{1}{x^2-y}$ __E__

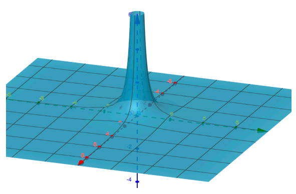
A)



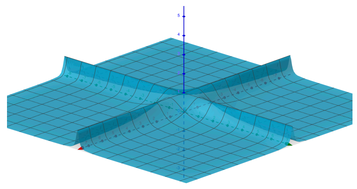
B)



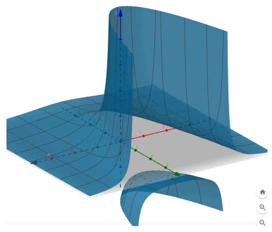
C)



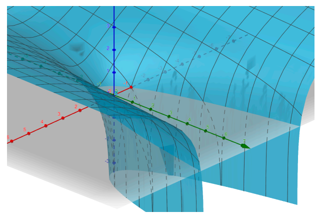
D)



E)



F)



(10) Match the following functions with their graphs: (12 points) (axes are in standard orientation with the x axis in red, y green, z blue.)

a) $f(x,y)=e^{x-y}$ __B__

(b) $f(x,y)=\cos(x)\cos(y)$ __C__

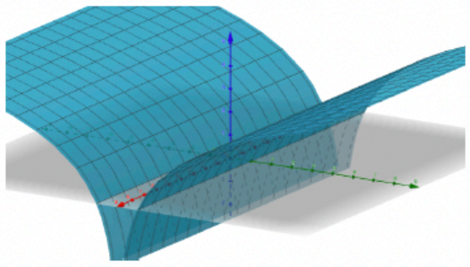
c) $f(x,y)=\ln(y^2)$ __A__

(d) $f(x,y)=\cos(xy)$ __D__

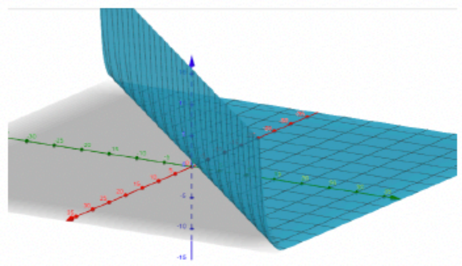
e) $f(x,y)=\dfrac{1}{x^2-y}$ __F__

(f) $f(x,y)=\dfrac{1}{x^2+y^2}$ __E__

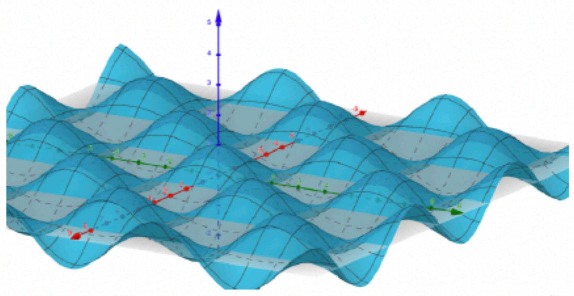
A)



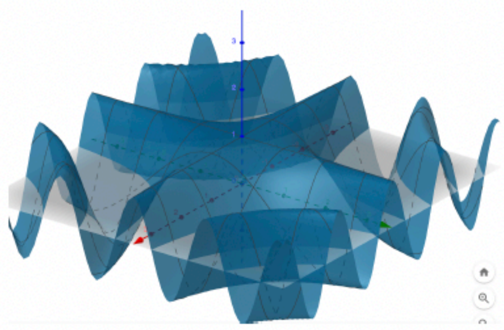
B)



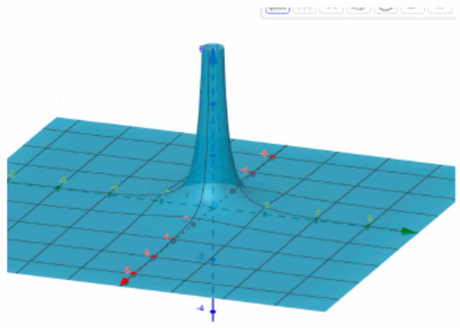
C)



D)



E)



F)